DEFINITION: algebraic multiplicity of the eigenvalue λ is the multiplicity of λ as a root of the characteristic equation $p(\lambda) = 0$.

DEFINITION: geometric multiplicity of the eigenvalue λ is the number of linearly independent eigenvectors associated with λ.

Ex Consider $Q = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Find $p(\lambda)$ and determine the **algebraic multiplicity** of each the eigenvalue.

$$P(\lambda) = \det \begin{pmatrix} 4-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = (4-\lambda)\left[(3-\lambda)(1-\lambda)\right] - 0 + 2\left[(3-\lambda)\right]$$

$$P(\lambda) = (3-\lambda)\left((4-\lambda)(1-\lambda)+2\right)$$

$$= (3-\lambda)^2(2-\lambda)$$

→ Algebraic Multiplicity = 1

→ Algebraic Multiplicity = 2

For $\lambda = 3$ Solve $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow v_1 - 2v_3 = 0 \Rightarrow v_1 = 2v_3$

v_2 anything

$$v_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2v_3 \\ v_2 \\ v_3 \end{pmatrix}$$

2 linearly Independent Eigenvectors associated with $\lambda = 3$

(b) Determine the **geometric multiplicity** of each eigenvalue.

For $\lambda_3 = 2 \Rightarrow \vec{v}_3$

the general solution is

Find v_3 using same method as above

$$\vec{y}(t) = c_1 e^{3t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} v_3$$

DEFINITION: Let A be an $n \times n$ matrix. If A has n eigenvectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ satisfying $\det[\vec{v}_1 \ \ \vec{v}_2 \ \ \cdots \ \ \vec{v}_n] \neq 0$, we say that A has a **full set of eigenvectors.**

DEFINITION: Let A be an $n \times n$ matrix. If A has at least one eigenvalue λ with $GM(\lambda) < AM(\lambda)$, this is A cannot have a full set of eigenvectors, then we call A **DEFECTIVE**.

Thm 3. Let A be an $n \times n$ matrix. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the distinct eigenvalues of A. Then,

$$A \ \text{ has a full set of eigenvectors} \Leftrightarrow \mathrm{AM}(\lambda_i) = \mathrm{GM}(\lambda_i) \quad \text{for each} \quad i = 1, 2, \cdots, k$$

Sec 4.7: Defective Matrices (geometric multiplicity < algebraic multiplicity)

What is the general solution to $\vec{Y}\,' = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \vec{Y}$?

Algorithm: Only for 2×2 matrices.
Let A be a 2×2 matrix that has only one (repeated) eigenvalue whose geometric multiplicity is 1. Let λ be this eigenvalue. In this case, do the following steps.
(1) Find the eigenvector $\vec{v}$ associated to the eigenvalue λ.
(2) Find a solution to the system

$$(A - \lambda I) \cdot \vec{w} = \vec{v}.$$

(3) A fundamental matrix for $\vec{Y}\,' = A \cdot \vec{Y}$ is $\Phi(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) \end{bmatrix}$ where $\phi_1(t)$ and $\phi_2(t)$ are given by

$$\phi_1(t) = \mathrm{e}^{\lambda t} \vec{v} \quad \text{ and } \quad \phi_2(t) = t\mathrm{e}^{\lambda t} \vec{v} + \mathrm{e}^{\lambda t} \vec{w}.$$

Ex1. Solve the i.v.p.

$$\vec{Y}\,' = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \vec{Y}, \qquad \vec{Y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$p(\lambda) = \begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix} = (2-\lambda)(4-\lambda)+1$

$\lambda^2 - 6\lambda + 9 = (\lambda-3)^2 \Rightarrow AM(3) = 2$

$GM(3)$ Solve $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad u_1 + u_2 = 0$, $u_1 = -u_2$ $\quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

~~Solution:~~ Ex. 2

$\vec{y}' = A\vec{y}$ $\quad A = \begin{bmatrix} 6 & 1 \\ -1 & 4 \end{bmatrix}$ $\quad p(\lambda) = \det\begin{pmatrix} 6-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} = (6-\lambda)(4-\lambda)+1 = (\lambda-5)^2$

$AM(5) = 2$ double root of $p(\lambda)$

Eigenvectors Solve $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ $v_1 + v_2$ $v_1 = -v_2$ $\quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad GM(5) = 1$

$0 < GM(5) < AM(5)$
Matrix is defective

Solve for generalised Eigenvector $\vec{w}$:

$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$ $w_1 + w_2 = 1 \Rightarrow w_1 = 1 - w_2 \Rightarrow \vec{w} = \begin{pmatrix} 1-w_2 \\ w_2 \end{pmatrix}$ → Infinitely many possibilities for a matrix

$\vec{y}(t) = c_1\begin{pmatrix} e^{5t} \\ -e^{5t} \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} t+1 \\ -t \end{pmatrix}$ $\quad \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$w_2 = 0 \ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$w_2 = 1 \ \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
... etc.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{y}(0) = \begin{pmatrix} c_1 + c_2 \\ -c_1 \end{pmatrix}$ $\quad c_2 = 1$
$c_1 = 0$

Thm 2 [Recognizing a Fundamental Matrix] Suppose A ($n \times n$ matrix) has a full set of eigenvectors. That is, if

$$\begin{array}{lll} \lambda_{\mathbf{1}} & \text{provides:} & \vec{v}_{\mathbf{1},1},\ \vec{v}_{\mathbf{1},2},\ \vec{v}_{\mathbf{1},3}, \cdots \vec{v}_{\mathbf{1},r_1} \quad \text{linearly independent eigenvector(s)} \\ \lambda_{\mathbf{2}} & \text{provides:} & \vec{v}_{\mathbf{2},1},\ \vec{v}_{\mathbf{2},2},\ \vec{v}_{\mathbf{2},3}, \cdots \vec{v}_{\mathbf{2},r_2} \quad \text{linearly independent eigenvector(s)} \\ & & \cdots \\ \lambda_{\mathbf{k}} & \text{provides:} & \vec{v}_{\mathbf{k},1},\ \vec{v}_{\mathbf{k},2},\ \vec{v}_{\mathbf{k},3}, \cdots \vec{v}_{\mathbf{k},r_k} \quad \text{linearly independent eigenvector(s),} \end{array}$$

where $r_i = \mathrm{GM}(\lambda_i) = \mathrm{AM}(\lambda_i)$ for each $i = 1, 2, \ldots, k$. Then, the matrix

$$\Phi(t) = \left[\mathrm{e}^{\lambda_1 t}\vec{v}_{\mathbf{1},1} \quad \mathrm{e}^{\lambda_1 t}\vec{v}_{\mathbf{1},2} \quad \cdots \quad \mathrm{e}^{\lambda_1 t}\vec{v}_{\mathbf{1},r_1} \quad \cdots\cdots \quad \mathrm{e}^{\lambda_k t}\vec{v}_{\mathbf{k},1} \quad \mathrm{e}^{\lambda_k t}\vec{v}_{\mathbf{k},2} \quad \cdots \quad \mathrm{e}^{\lambda_k t}\vec{v}_{\mathbf{k},r_k} \right]$$

is a fundamental matrix for $\vec{Y}\,' = A \cdot \vec{Y}$.

Important remarks:

- $\mathbf{k}$ is the number of distinct eigenvalues of the matrix A. The number $\mathbf{k}$ may not be n.
- Theorem 1 is the particular case when $\mathbf{k} = n$.